GeoGebra Tools with Proof Capabilities

Zoltán Kovács and Csilla Sólyom-Gecse

STEM Education Centre
Johannes Kepler University of Linz, Austria

Sixth Central and Eastern European Conference on
Computer Algebra and *Dynamic Geometry Systems*
in *Mathematics Education*

Târgu Mureș, Romania, 7 September 2016

Abstract

We report about significant enhancements of the complex algebraic geometry theorem proving subsystem in GeoGebra for automated proofs in Euclidean geometry, concerning the extension of numerous GeoGebra tools with proof capabilities. As a result, a number of elementary theorems can be proven by using GeoGebra's intuitive user interface on various computer architectures including native Java and web based systems with JavaScript. We also provide a test suite for benchmarking our results with more than 200 test cases.

Outline

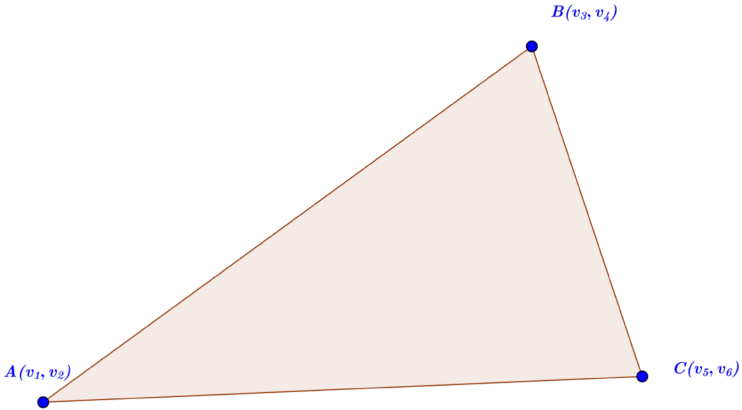
Introduction

- What is GeoGebra?
- Automated theorem proving (ATP) and computer algebra systems (CAS).
- Using Gröbner bases.

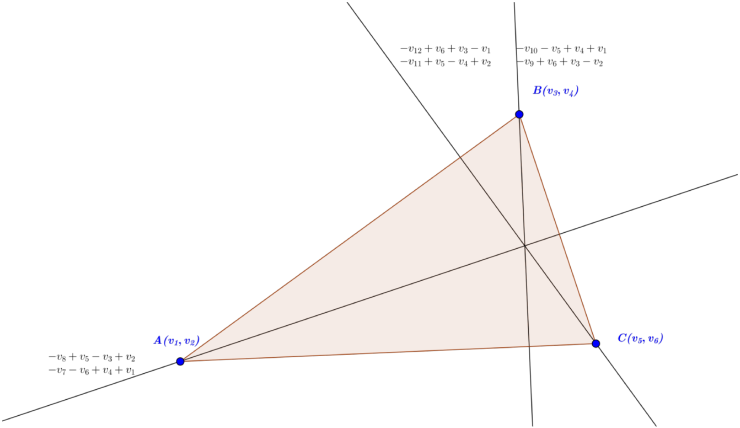
The ProveDetails algorithm (Recio-Vélez, 1996, extended)

1. Turn the input question $\mathcal{Q} = (H_1, \ldots; T)$ to an eq. sys. S with vars $v_1, \ldots$ by using reductio ad absurdum, Rabinowitsch's trick and the fixed point $A(0,0)$.
2. Send S with some order of the indep. vars as input to the CAS (Giac/Singular) for computing an equivalent equation system by eliminating the dep. vars.
3. Get an equivalent equation system S' back as set of equations with polynomial factors on the LHS and 0 on the RHS. ("Flatten" factors with higher exponents.)
4. If $S' = \{1 = 0\}$, then output: "always true".
5. If $S' = \{0 = 0\}$, then output: "false".
6. Otherwise, for each polynomial p of the LHS of eq. $p = 0$ in S',
    1. let score $s_p := 0$ initially,
    2. for each factor f in p,
        - if f has a usable geometrical meaning, then add the educational usability score u_f to s_p, else remove $p = 0$ from S'.
7. If $S' = \{x_P - x_Q = 0, y_P - y_Q = 0\}$ for the variables of the free points P and Q, then output: "true if $P \neq Q$".
8. Choose the best polynomial p in S' with the lowest score $s_p < \infty$, output: the geometrical meanings of its factors in denied form as conjectures.
9. Otherwise output: "true under certain conditions".

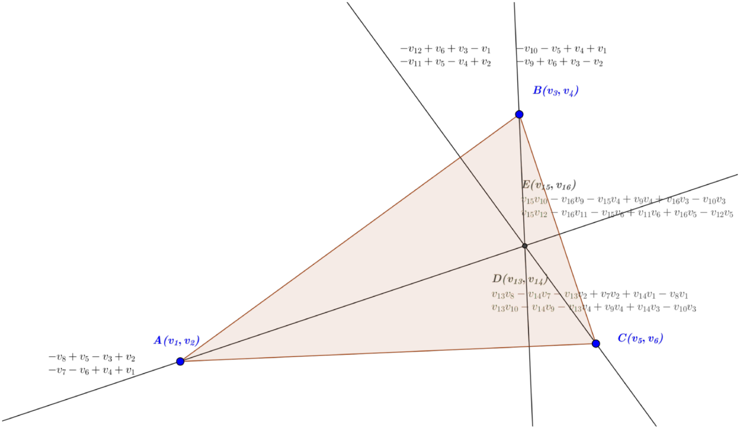
Setting up the free variables



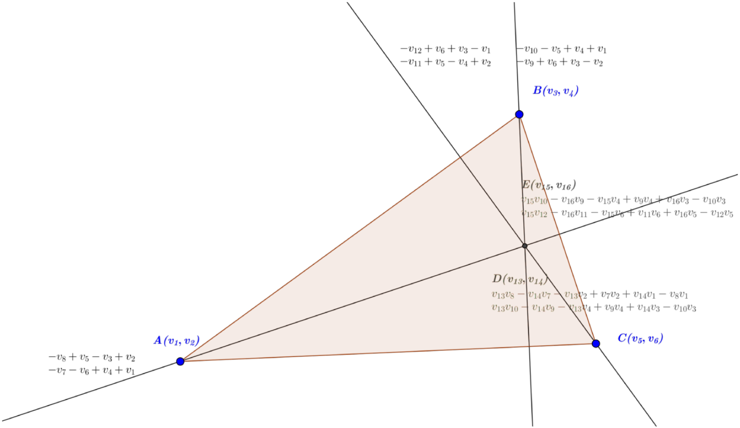
Setting up polynomial equations for the hypotheses



Setting up polynomial equations for the hypotheses



Setting up the conjecture (thesis) in denied form



$$((v_{13} - v_{15}) \cdot v_{17} - 1) \cdot ((v_{14} - v_{16}) \cdot v_{17} - 1) = 0$$

Eliminating the non-free variables from the eq. system

```
Welcome to giac readline interface
(c) 2001,2015 B. Parisse & others
Homepage http://www-fourier.ujf-grenoble.fr/~parisse/giac.html
Released under the GPL license 3.0 or above
See http://www.gnu.org for license details
May contain BSD licensed software parts (lapack, atlas, tinymt)
-------------------------------------------------
Press CTRL and D simultaneously to finish session
Type ?commandname for help
0>> _
```

Eliminating the non-free variables from the eq. system

```
0>> factor(eliminate(subst([-v12+v6+v3-v1,-v11+v5-v4+v2,
-v10-v5+v4+v1,-v9+v6+v3-v2,-v8+v5-v3+v2,-v7-v6+v4+v1,
v15*v10-v16*v9-v15*v4+v9*v4+v16*v3-v10*v3,
v15*v12-v16*v11-v15*v6+v11*v6+v16*v5-v12*v5,
v13*v8-v14*v7-v13*v2+v7*v2+v14*v1-v8*v1,
v13*v10-v14*v9-v13*v4+v9*v4+v14*v3-v10*v3,
((v13-v15)*v17-1)*((v14-v16)*v17-1)],[v1=0,v2=0]),
[v7,v8,v9,v10,v11,v12,v13,v14,v15,v16,v17]))
[v5*v4-v3*v6,
v6*(v5*v3-v3^2+v6*v4-v4^2),
-v3*(v4*v6+v3*v5-v5^2-v6^2),
v6*(v4-v6)*(v3^2+v4^2)]
// Time 0.03
```

Translating the non-degeneracy conditions back to geometry

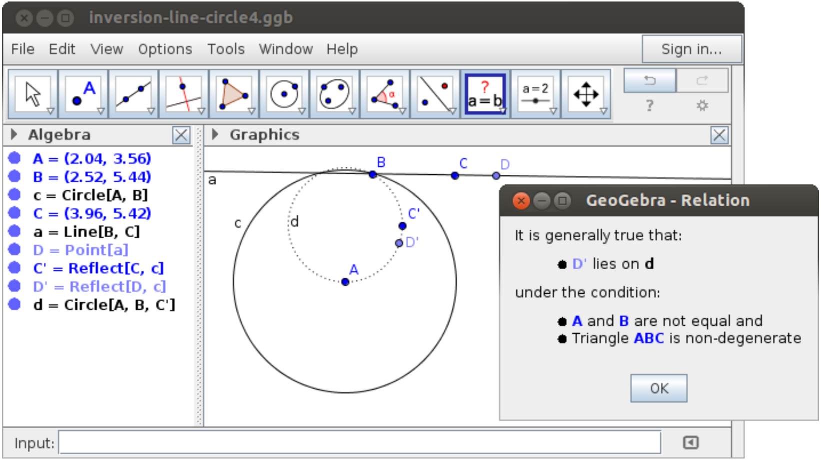
From the "Gröbner basis" black box:

1. $v_5 v_4 - v_3 v_6 = 0$,
2. $v_6 \cdot (v_5 v_3 - v_3^2 + v_6 v_4 - v_4^2) = 0$,
3. $-v_3 \cdot (v_4 v_6 + v_3 v_5 - v_5^2 - v_6^2) = 0$,
4. $v_6 \cdot (v_4 - v_6) \cdot (v_3^2 + v_4^2) = 0$.

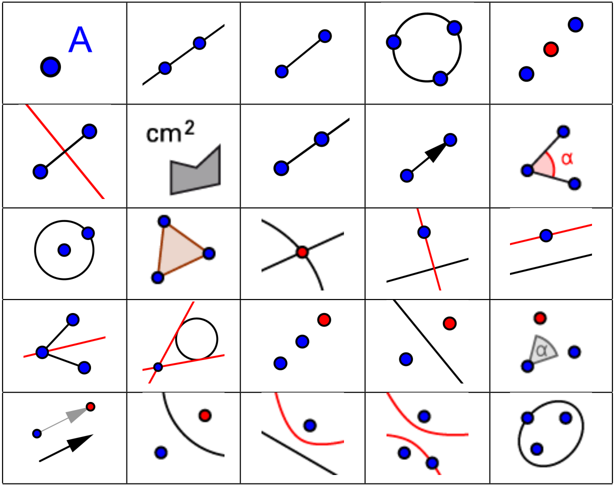
Geometric translations:

1. Let ABC be a non-degenerate triangle. Then the intersection points of the altitudes are identical.
2. Let ABC be a triangle with $AB \not\perp AC$ such that the y-coordinates of A and C differ. Then the intersection points of the altitudes are identical.
3. Let ABC be a triangle with $AC \not\perp BC$ such that the x-coordinates of A and B differ. Then...
4. Let ABC be a triangle such that the y-coordinates of A and C, and B and C, differ, and also $A \neq B$. Then...

Relation Tool demo



Implemented tools



ProveDetails algorithm: a comparison (2015)

Test name | GB (Singular) | GB (Giac) | Wu (OpenGeoProver) | Auto

Test name	GB (Singular)	GB (Giac)	Wu (OpenGeoProver)	Auto
Simson2	$A=B$ $A=C$ $B=C$ 1275	5921	$b=h_1$ 232	$b=h_1$ 6122
square1	**45**	*256*	83	255
square2	**48**	*268*	. . . 112	257
square3	**46**	*270*	104	256
symmedians	19	230	75	295
Thales1	$A=C$ 52	$A=C$ 280	$f_1=g$ 193	$A=C$ 265
Thales2	$A=B$ 55	$A=B$ 295	**113**	$A=B$ 311
Thales3	$A=C$ 53	$A=C$ 291	$B=C$ 120	$A=C$ 287
triangle-areas	52	290	63	390
triangle-m. . .ns	**42**	*271*	$a=f_1$ $d \parallel e$ 475	255
triangle-m. . .t1	**38**	*244*	77	248
triangle-m. . .t2	**38**	*260*	76	244
triangle-m. . .t3	**41**	*246*	86	256
triangle-m. . .t4	**30**	*267*	78	269
triangle-m. . .t5	**37**	*243*	83	249
true	2	**1**	*2*	1
Varignon	**40**	*248*	85	266
Total (of 60)	**47**	**45**	**48**	**49**

Current status: `http://tinyurl.com/provertest`
(246 test cases as of 2016 August)

Use in education?

. . .increased availability in school mathematics instruction of . . . dynamic geometry systems. . . raised the concern that such programmes would make the boundaries between conjecturing and proving even less clear for students. . . [They] allow students to check easily and quickly a very large number of cases, thus helping students "see" mathematical properties more easily and potentially "killing" any need for students to engage in actual proving. (Lin, Yang, Lee, Tabach and G. Stylianides, "Principles of Task Design for Conjecturing and Proving", Springer, 2012, 305–326.)

References

- G. Ancsin, M. Hohenwarter & Z. Kovács (2013). **GeoGebra goes web.** *The Electronic Journal of Mathematics and Technology*, 7(6):412–418.
- M. Hohenwarter, M. Borcherds, G. Ancsin, B. Bencze, M. Blossier, A. Delobelle, C. Denizet, J. Éliás, Á. Fekete, L. Gál, Z. Konečný, Z. Kovács, S. Lizelfelner, B. Parisse & G. Sturr (2014). **GeoGebra 5.**
- Z. Kovács & B. Parisse (2015). **Giac and GeoGebra — improved Gröbner basis computations.** In J. Gutierrez, J. Schicho & M. Weimann (eds.), *Computer Algebra and Polynomials*, Lecture Notes in Computer Science, p. 126–138. Springer.
- F. Botana, M. Hohenwarter, P. Janičić, Z. Kovács, I. Petrović, T. Recio & S. Weitzhofer (2015). **Automated theorem proving in GeoGebra: Current achievements.** *Journal of Automated Reasoning*, 55(1):39–59.
- Z. Kovács (2015). **The Relation Tool in GeoGebra 5.** In *Post-conference Proceedings of the 10th International Workshop on Automated Deduction in Geometry (ADG 2014), 9-11 July 2014*, Lecture Notes in Computer Science. Springer.
- Z. Kovács & C. Sólyom-Gecse (2016): **GeoGebra Tools with Proof Capabilities**. *arXiv*:1603.01228v1 [cs.AI]